TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 2

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Homework 2.1 (A convergence criterion)**.** Let $D \subset \mathbf{C}$ be a domain and let $(f_n)_{n\in\mathbf{N}}$ be a locally uniformly bounded sequence of holomorphic functions $f_n \colon D \to \mathbf{C}$. Assume there exists $z_0 \in D$ such that for every $k \in \mathbf{N}_0$, the following limit exists:

$$\lim_{n\to\infty} f_n^{(k)}(z_0).$$

Show $(f_n)_{n\in\mathbf{N}}$ converges locally uniformly to some function $f\colon D \to \mathbf{C}$[1].

Homework 2.2 (A compactness criterion)**.** Let $B_1(0) \subset \mathbf{C}$ denote the open unit disk. Define the family of functions

$$\mathcal{F} = \Big\{ f\colon B_1(0) \to \mathbf{C} : f \text{ holomorphic},\ f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ for every } z \in B_1(0),$$
$$|a_k| \leq 1 \text{ for every } k \in \mathbf{N}_0 \Big\}.$$

Let $(f_n)_{n\in\mathbf{N}}$ be an arbitrary sequence in $\mathcal{F}$.

a. Show $(f_n)_{n\in\mathbf{N}}$ subconverges locally uniformly to a function $f\colon B_1(0) \to \mathbf{C}$.
b. Show f belongs to $\mathcal{F}$.

Homework 2.3 (An extremal problem in the proof of the Riemann mapping theorem*)**.** Let $D \subsetneq \mathbf{C}$ be a simply connected domain containing zero. Show there exists a holomorphic function $f\colon D \to \mathbf{C}$ which attains the supremum

$$s_0 := \sup\{|f'(0)| : f\colon D \to B_1(0),\ f \text{ holomorphic and injective},\ f(0) = 0\}.$$

We consider a sequence $(f_n)_{n\in\mathbf{N}}$ of functions $f_n \colon D \to B_1(0)$ which are admissible in the previous supremum[2] such that $|f_n'(0)| \to s_0$ as $n \to \infty$.

a. Show $(f_n)_{n\in\mathbf{N}}$ subconverges to a holomorphic function $f\colon D \to B_1(0)$.
b. Show s_0 is a real number.
c. Show the above supremum is attained for the function f from a. (in particular, f is admissible in the definition of s_0).

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[1] **Hint.** Establish pointwise convergence first.

[2] The nonemptiness of the class of functions satisfying the desired properties is indeed not obvious. You may assume this fact as given without proof (which will be given later during the lecture, based on D being simply connected).

Homework 2.4 (Locally normal vs. locally uniform convergence)**.** Let $(f_j)_{j\in\mathbf{N}}$ be a sequence of functions $f_j\colon U \to \mathbf{C}$, where $U \subset \mathbf{C}$ is open. The series $\sum_{j=1}^{\infty} f_j$ will be called *locally normally convergent* if for every $z_0 \in U$ there exists $r > 0$ such that

$$\sum_{j=1}^{\infty} \sup_{z\in B_r(z_0)} |f_j(z)| < \infty.$$

a. Show if $\sum_{j=1}^{\infty} f_j$ is locally normally convergent, then this series is locally uniformly convergent.
b. Give an example of a sequence $(f_j)_{j\in\mathbf{N}}$ of not necessarily holomorphic functions $f_j\colon \mathbf{C} \to \mathbf{C}$ which shows the converse of a. is in general false.